Friendly workshop on Diophantine Equations and related problems
(FWDERP 2019, Bursa Uludağ University, Bursa, Turkey)

Problem Session

edited by Alain Togbé

Problem 1. *(by Prof. Refik Keskin)*
We have the following conjecture

Conjecture 2. *Let m and n be positive integers. Then the Diophantine equation*

$$m^2x^2 + 4xy - n^2y^2 = 4 \tag{1}$$

has a positive solution (x, y) if and only if $m = n$ or $m = 1, 2$, or $n = 1, 2$.

Problem 3. *(by Prof. Refik Keskin)*
Can you characterize all solutions of the Diophantine equation

$$x^2 + kxy - y^2 = k, \tag{2}$$

where k is an integer.

Problem 4. *(by Prof. Kálmán Győry)*
Characterize all number fields $\mathbb{K}$ such that

$$\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[\alpha] = \mathbb{Z}[1, \alpha, \alpha^2, \cdots, \alpha^{d-1}], \tag{3}$$

where $d = [\mathbb{K} : \mathbb{Q}]$ and α is a unit in $\mathcal{O}_{\mathbb{K}}$. [It is analogue to Hasse's problem concerning monogenic number fields].

Problem 5. *(by Prof. Kálmán Győry)*
Given d, denote by $N_d(X)$ the number of number fields of degree d such that $\mathcal{O}_{\mathbb{K}}$ is generated over $\mathbb{Z}$ by a single unit of $\mathcal{O}_{\mathbb{K}}$ and $|D_{\mathbb{K}}| \leq X$. Is it true that $N_d(X)$ tends to infinity as X tends to infinity? [This is closely related to a recent quantitative theorem of Barghava and al.]

Problem 6. *(by Prof. Andrzej Dąbrowski)*
Is the set of integer solutions of the equation

$$n! + 1 = x^2 + y^2 \tag{4}$$

infinite ? (For a short discussion see section 3 in the paper: A. Dąbrowski, On the Brocard-Ramanujan problem and generalizations, Coll. Math. **126** *(2012), 105-11).*

Problem 7. *(by Prof. Andrzej Dąbrowski)*
Let $\mathcal{C}$ denote the classical Cantor set. Consider fractional parts of the values of the Riemann zeta function $\zeta(s)$ at integral arguments $n > 1$ (we can consider integers $n \neq 1$). Determine the set of n's such that $\{\zeta(n)\} \in \mathcal{C}$. Similar question for L-function attached to an elliptic curve over a number field K (or attached to a modular form or to a motive).

Problem 8. *(by Prof. Shanta Laishram)*
Let

$$S = \{n \in \mathbb{N} : \ \sigma_2(n) = \sigma_3(n)\}, \tag{5}$$

where $\sigma_k(n)$ denotes the sum of digits of n in base k, ($k \geq 2$ integer.)
Is S infinite?

Equivalently, solve the equation

$$2^{n_0} + 2^{n_1} + \cdots 2^{n_s} = \sum_{i=0}^{r} a_i 3^{m_i}, \tag{6}$$

where $a_i \in \{1, 2\}$.

Problem 9. *(by Prof. Shanta Laishram)*
It is known that the equation

$$\prod_{i=1}^{n} (i^2 + 1) = y^2 \tag{7}$$

has only the solution given by $2 \cdot 5 \cdot 10 = 10^2$.

Conjecture 10. *The equation*

$$\prod_{i=1}^{n} ((n + i)^2 + 1) = y^2, \tag{8}$$

in variables n, k and y has only the solution given by $2 \cdot 5 \cdot 10 = 10^2$.

Problem 11. *(by Prof. Eva Goedhard)*
Bugeaud and Dujella [2]–[3] considered the problem of finding positive integers a and b such that one of the sets $\{a+1, b+1, ab+1\}$ or $\{a+1, ab+1, ab^2+1\}$ is comprised entirely of perfect k-th powers, for $k \geq 3$. (The number of solutions is known to be finite as it was considered by A. Kihel and O. Kihel [5] for more general sets). This problem consists of solving

$$(x^k - 1)(y^k - 1) = z^k - 1 \tag{9}$$

or

$$(x^k - 1)(y^k - 1) = (z^k - 1)^2. \tag{10}$$

Bugeaud [2] and Bennett [1] gave all solutions of equation (9) for $k \geq 3$ and characterized those of equation (10), for $k \geq 4$.

How many solutions are there to equation (10) for $k = 3$?

Problem 12. *(by Prof. Eva Goedhard)*
Let x, y, $z \geq 1$ be integers and a, b, c be fixed positive integers. The equation

$$(a^2cx^k - 1)(b^2cy^k - 1) = (abcz^k - 1)^2 \tag{11}$$

has no solutions for $a^2x^k \neq b^2y^k$ when $k \geq 7$.

a. What happens when $k = 3, 4, 5, 6$?

b. Solve

$$(a^2cx^3-1)(b^2cy^3-1)=z^2 \tag{12}$$

when $a^2x^3 \neq b^2y^3$.

References

[1] M.A. Bennett, "The Diophantine equation $(x^k-1)(y^k-1)=(z^k-1)^t$", *Indag. Math. (N.S.)*, **18** (2007), no. 4, 507–525.

[2] Y. Bugeaud, "On the Diophantine equation $(x^k-1)(y^k-1)=(z^k-1)$", *Indag. Math. (N.S.)*, **15** (2004), no. 1, 21–28.

[3] Y. Bugeaud and A. Dujella,"On a problem of Diophantus for higher powers", *Math. Proc. Cambridge Philos. Soc.*, **135** (2003), no. 1, 1–10.

[4] E. Goedhart and H.G. Grundman, "On the Diophantine equation $NX^2+2^L3^M=Y^N$", *J. Number Theory* **141** (2014), 214–224.

[5] A. Kihel and O. Kihel, "Sets in which the product of any K elements increased by t is a kth-power", *Fibonacci Quart.* **39** (2001), no. 2, 98–100.

Problem 13. *(by Prof. Huilin Zhu)*
For example, let us consider the set

$$S_0=\{a_1=2^3-1, \quad a_2=4^3-1, \quad a_3=22^3-1\}.$$

Notice that the a_i *verify*

$$a_1\cdot a_2=21^2, \quad a_1\cdot a_3=819^2, \quad a_2\cdot a_3=2457^2.$$

and the cardinal number of S_0 *is* 3.

- *In general, we consider the set* S *of the numbers* a_i *defined by*

$$a_i=x_i^n-1=d{m_i}^2,$$

where $x_i>1$, $n>2$ *is a constant and* d *is a square-free constant. What is the cardinal of* S*? Could it be infinite? If not, what is the bound of its cardinality?*

- *More generally, if* n *is not fixed, i.e.*

$$a_i=x_i^{n_i}-1=d{m_i}^2,$$

where $x_i>1$, $n_i>2$ *and* d *is square-free constant. Then what can we say about the cardinality of the set* S*?*

Problem 14. *(by Prof. Omar Kihel)*
Let $\mathbb{K}$ *be a number field and* R *its ring of integers. For every* $\alpha\in R$*, does exist a primitive element* $\theta\in R$ *such that* $\alpha\in\mathbb{Z}[\theta]$*?*

Problem 15. *(by Prof. Omar Kihel)*
Find an example of a number field $\mathbb{K}$ *such that*

$$R=\cup_{i=1}^n\mathbb{Z}[\theta_i], \tag{13}$$

i.e. its ring of integers R *is not monogenic.*

The remaining problems of this document are proposed by Prof. Claude Levesque and Prof. Michel Waldschmidt

Let $\mathbb{K}$ be a number field. Using Schmidt's Subspace Theorem, it has been proved in [2, Corollary 3.6] that there exists a positive constant $c_{\mathbb{K}}$, depending only on $\mathbb{K}$ with the following property. Let ε be a unit in $\mathbb{K}$ of degree ≥ 3. Let $F_\varepsilon(X,Y) \in \mathbb{Z}[X,Y]$ be the binary form of degree $[\mathbb{Q}(\varepsilon):\mathbb{Q}]$ such that $F_\varepsilon(X,1)$ is the irreducible polynomial of ε. If the height $h(\varepsilon)$ of ε satisfies $h(\varepsilon) \geq c_K$, then there is no $(x,y) \in \mathbb{Z}^2$ satisfying $F_\varepsilon(x,y) = \pm 1$ and $xy \neq 0$.

Here, h is the logarithmic height - since two norms on a finite dimensional vector space are equivalent, the choice of the height does not matter.

Our first problem is the following one.

Problem LW1. *Give an effective upper bound for c_K.*

One could be more ambitious and ask for an upper bound depending only on the degree of $\mathbb{K}$. For such a uniform estimate, some exceptional families of Thue equations having very large exceptional solutions should be excluded... at least to start with. Let us give examples of such families.

Let $c \in \{1, -1\}$. Let $g(X)$ be a non constant monic polynomial of degree $n-2$ and let a be a rational integer. Consider the (assumed irreducible) polynomial

$$f(X) = X(X-a)g(X) + c,$$

which is the minimal polynomial of a unit of degree n . Then the binary form

$$F(X,Y) = Y^n f(X/Y) = X(X-aY)Y^{n-2}g(X/Y) + cY^n$$

satisfies $F(a,1) = c,\ F(-a,-1) = (-1)^n c$. This shows that the exceptional solution $(a,1)$ can be as large as one pleases.

Consider the special case where the polynomial $(X-a)g(X)$ has $n-1$ distinct roots in $\mathbb{Z}$. Let $c \in \{1,-1\}$, $n \geq 3$ and let $f(X)$ be the (assumed irreducible) polynomial

$$f(X) = X \prod_{i=1}^{n-1} (X - a_i) + \ c$$

where the a_i's are distinct nonzero integers. Then the binary form

$$F(X,Y) = Y^n f(X/Y) = X \prod_{i=1}^{n-1} (X - a_i Y) \ + cY^n$$

satisfies for $i = 1, \ldots, n-1$,

$$F(a_i, 1) = c,\ F(-a_i, -1) = (-1)^n c.$$

This last example shows that the larger n is, the larger the cardinality of exceptional solutions is.

As a first step towards a uniform bound, we restrict ourselves to specific families of Thue equations. To start with, consider the family $\mathbb{K}_t$ $(t \geq 0)$ of the simplest cubic fields of Shanks. The field $\mathbb{K}_t$ is the cyclic cubic field $\mathbb{Q}(\upsilon_t)$ where υ_t is a root of the polynomial $P_t(X,1)$ where

$$P_t(X,Y) = X^3 - (t-1)X^2Y - (t+2)XY^2 - Y^3.$$

Problem LW2. *Does there exist an absolute positive constant c with the following property. Let $t \geq 0$ and let ε be a unit of K_t with $\varepsilon \neq \pm 1$. Let $(x,y) \in \mathbb{Z}\times\mathbb{Z}$ satisfy $\max\{|x|,|y|\} \geq 2$ and $F_\varepsilon(x,y) = \pm 1$. Then*

$$\max\{|x|,|y|,t,h(\varepsilon)\} \leq c. \tag{14}$$

For this family, a partial result is known, involving a rank one subgroup of the group of units of K_t. Indeed, the following result is proved in [3]. The assumptions $a \geq 1$, $F_{u_t^a}(x,y) = \pm 1$ and $\max\{|x|,|y|\} \geq 2$ imply (14) with $\varepsilon = u_t^a$ and c replaced with another effectively computable absolute constant. It would be interesting to decide whether the list of 27 exotic solutions given in [3, § 15] is complete.

Problem LW3. *Is it true that the conditions $F_{u_t^a}(x,y) = \pm 1$ and $\max\{|x|,|y|\} \geq 2$ imply $t \leq 4$ and $a \leq 5$?*

The next and last problem is a program of research. Let $F_t \in \mathbb{Z}[X,Y]$ be a family of irreducible binary forms of degrees ≥ 3. For each $t \geq 0$, let α_t be a root of $F_t(X,1)$. For ε a unit of degree ≥ 3 in $\mathbb{Q}(\alpha_t)$, we have a Thue equation $F_\varepsilon(X,Y) = \pm 1$ for which it makes sense to consider the analogs of Problems LW2 and LW3.

Problem LW4. *For each of the 22 families of Thue equations listed in* [1, §4.1]*, prove that there exists an absolute constant $c > 0$ such that, as soon as $\max\{t, h(\epsilon)\} \geq c$, there is no $(x,y) \in \mathbb{Z}^2$ satisfying $\max\{|x|,|y|\} \geq 2$ and $F_\varepsilon(x,y) = \pm 1$.*

A first step would be to consider only a rank one subgroup of the units of $\mathbb{Q}(\alpha_t)$. So far, this has been done only for the simplest cubic fields.

References

[1] C. Heuberger, *Parametrized Thue Equations. A Survey* , Proceedings of the RIMS symposium Analytic Number Theory and Surrounding Areas, Kyoto, Oct 18–22, 2004, RIMS Kôkyûroku, vol. 1511, 2006, pp. 82–91.
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[2] C. Levesque and M. Waldschmidt, *Familles d'équations de Thue–Mahler n'ayant que des solutions triviales*, Acta Arith. **155** (2012), 117–138.
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[3] C. Levesque and M. Waldschmidt, *A family of Thue equations involving powers of units of the simplest cubic fields*, J. Théor. Nombres Bordx. **27**, No. 2 (2015), 537–563.
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